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Papers

An Approximate Solution for Fredholm Integral Equation of the Second Kind in the Space $L^2_{p(x)}\ [0, 2\pi]$ with Weight Function $p(x)$

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Abstract— **In the present paper we study the approximate solution for Fredholm integral equation of the second kind in the space L2p(x)[0,2π] with weight function p (x) ≥ 1 and bounded almost every where on [0,2π]. The technique of this study is based on linear polynomial operators Un (φ; x) which generate good approximation to the function φ (x) in the space L2p(x)[0,2π], where the given equation is replaced by Fredholm integral equation with degenerate kernel. The solution of the new equation is taken as an approximate solution to the original equation, and also we give estimates of the errors which arise in this connection. This approximation is discussed in details for Dirichlet, Vallee-Poussin, Féjer, Rogozinski and Jackson operators.**

Index Terms— **Solution of Fredholm integral equation of the second kind, Linear polynomial operators, space with weight function.**

I. INTRODUCTION

The approximate solution of linear integral equations have been studied by many authors in the literature; see [1], [4], [8]. In this paper, we consider the following Fredholm integral equation of the second kind,

$$\varphi(x) = f(x) + \lambda \int_0^{2\pi} k(x,y)\,\varphi(y)\,dy, \qquad (1)$$

in which all functions are 2π-periodic with respect to x and y, $f(x)$ is in $L^2_{p(x)}[0,2\pi]$, λ is some complex number such that $1/\lambda$ is a regular value of the kernel $k(x, y)$ and the kernel $k(x, y)$ satisfies the following conditions (A^*):

i) $$|\lambda|\, \| k(x,y) \|_{L^2_p} < 1, \qquad (2)$$

where

$$\| k(x,y) \|_{L^2_p} = \| k(x,y) \|_{L^2_p[Q]} =$$

$$= \left[\int_0^{2\pi}\int_0^{2\pi} p(x)\,p(y)\,[k(x,y)]^2\,dx\,dy \right]^{\frac{1}{2}},$$

$$Q = \{(x,y): 0 \le x, y \le 2\pi\}$$

ii) the functions:

$$A(x) = \left[\int_0^{2\pi} p(y)[k(x,y)]^2 dy\right]^{\frac{1}{2}},\ B(y) = \left[\int_0^{2\pi} p(x)[k(x,y)]^2 dx\right]^{\frac{1}{2}} \qquad (3)$$

are bounded almost everywhere by the number M,

$$ess\ sup\ A(x) \le M,\ \ ess\ sup\ B(y) \le M.$$

Instead of equation (1), we solve the following equation

$$\varphi_n(x) = U_n(f; x) + \lambda \int_0^{2\pi} U_n[k(.,y); x]\,\varphi_n(y)\,dy, \qquad (4)$$

the notation $U_n[k(., y); x]$ will mean that the operator U_n acts on $k(t, y)$ as a function of t, and at the same time, the variable y plays the role of a parameter.

Now, since the functions $U_n(f; x)$ and $U_n[k(., y); x]$ are both trigonometric polynomials of order n with respect to x, the solution $\varphi_n(x)$ of equation (4) will also be trigonometric polynomial of order n.

More precisely, it is shown in this paper that:

$$\| \varphi - \varphi_n \|_{L^2_p} \le (1 + \alpha_n(k))\, \| \varphi - U_n(\varphi; x) \|_{L^2_p}, \qquad \alpha_n \to 0$$

where

$$\| \varphi(x) \|_{L^2_{p(x)}} = \| \varphi(x) \|_{L^2_p} = \left[\int_0^{2\pi} p(x)\,|\varphi(x)|^2\,dx \right]^{\frac{1}{2}}.$$

II. PRELIMINARIES

The linear polynomial operators $U_n(g; x)$ which are good approximation to the function $g(x)$ in the space $L^2_{p(x)}$ have the form :

$$U_n(g;x) = \frac{1}{\pi}\int_0^{2\pi} g(t)\,U_n(x-t)\,dt = \frac{1}{\pi}\int_0^{2\pi} g(x-t)\,U_n(t)\,dt, \qquad (5)$$

where

$$U_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos(kx), \qquad (6)$$

$\lambda_k^{(n)}$ are given as constants depending on the linear method.

We shall recall the most important of these methods which are essential for our purposes [2], [3], [5], [9],

1. Dirichlet method $D_n(f; x)$ (the method of partial sums):

This method is obtained by letting $\lambda_k^{(n)} = 1$, $k = 1, 2, \ldots, n$, i.e., with the help of the Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = [\sin(\tfrac{1}{2}(2n+1)x)]/[2\sin(\tfrac{x}{2})].$$

2. Féjer method $F_n(f; x)$ (the method of arithmetic means):

This method is obtained by letting $\lambda_k^{(n)} = 1 - (k/n)$, $k = 1, 2, \ldots, n$, i.e., with the help of the Féjer kernel

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$$F_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\left[1-(k/n)\right]\cos(kx)=[\sin^2(nx/2)]/[2\sin^2(x/2)].$$

3. The method of Vallee-Poussin $V_n(f;x)$ is obtained with the help of the kernel

$$V_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos(kx)+\sum_{k=n+1}^{2n}\left[2-(k/n)\right]\cos(kx)=$$
$$=2F_{2n}(x)-F_n(x).$$

4. The method of Rogosinski $R_n(f;x)$ is obtained with the help of the kernel

$$R_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos(k\pi/2n)\cos(kx)=$$

$$=\frac{1}{2}[\sin(\pi/2n)\cos(nx)]/[\cos(x)-\cos(\pi/2n)].$$

5. The method of Jackson $J_n(f;x)$ is obtained with the help of the kernel

$$J_n(x)=\frac{1}{2}+\sum_{k=1}^{2n-2}\mu_k\cos(kx)=$$
$$=[3\sin^4(nx/2)]/[2n(2n^2+1)\sin^4(x/2)],$$

where the μ_k are numbers which we will not write out.

THEOREM 1. [3]
For any kernel $k(x,y)\in L^2_p[Q]$, if the linear polynomial operators U_n of order n is defined in $L^2_{p(x)}$ and if the function $f(x)\in L^2_{p(x)}$, then

$$U_n\left[\int_a^b k(.,y)f(y)dy\,;x\right]=\int_a^b U_n\left[k(.,y);x\right]f(y)dy.$$

THEOREM 2. [6]

If A and B are two bounded linear operators in Banach space E, while A has an inverse and $\|B\|_E\,.\,\|A^{-1}\|_E<1$ then the operator $(A+B)$ has also an inverse and

$$\|(A+B)^{-1}\|_E\le\|A^{-1}\|_E\,/\,(1-\|B\|_E\,\|A^{-1}\|_E).$$

THEOREM 3. [7]
For $f(x)$ and $k(x,y)$ belongs to L^2_p, if $|\lambda|\,\|k(x,y)\|_{L^2_p}<1$, then the integral equation

$$\varphi(x)=f(x)+\lambda\int_0^{2\pi}k(x,y)\varphi(y)dy,$$

has a unique solution $\varphi(x)$ in $L^2_{p(x)}$.

III. AUXILIARY DEFINITIONS AND THEOREMS

The discussion in the present paper are stimulated and achieved by the following procedure:

Definition 1.
The mean-modulus of continuity of the kernel $k(x,y)\in L^2_p$ is defined by the function

$$\omega_{L^2_p}(k;t)=\omega_{L^2_p}(t)=$$
$$=\frac{1}{2\pi}\max_{|s|\le t}\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[k(x-s,y)-k(x,y)]^2dxdy\right]^{\frac{1}{2}}.\qquad(7)$$

It is evident that:

1- $\omega_{L^2_p}(0)=0$, 2- $\omega_{L^2_p}(t_1+t_2)\le\omega_{L^2_p}(t_1)+\omega_{L^2_p}(t_2)$,

3- $\omega_{L^2_p}(t)$ is positive and monotone increasing function,

4- $\omega_{L^2_p}(\eta t)\le(1+\eta)\,\omega_{L^2_p}(t)$, for any positive real number η.

Definition 2.

The value of the following norm :

$$\delta_n(k)=\delta(k;U_n)=\|U_n(k(.,y);x)-k(x,y)\|_{L^2_p}=$$
$$=\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[U_n(k(.,y);x)-k(x,y)]^2dxdy\right]^{\frac{1}{2}},\qquad(8)$$

will play an important role in estimating the error arising from the replacement of equation (1) by equation (4).
The following theorem provides an estimate of $\delta(k;U_n)$.

Theorem 4.
For any square-summable kernel $k(x,y)\in L^2_p$ and for any linear polynomial operator $U_n(g;x)$, we always have the inequality :

$$\delta_n(k)\le 2\,\omega_{L^2_p}(k;\frac{1}{n})\int_0^{2\pi}|U_n(t)|(n|t|+1)dt.\qquad(9)$$

Proof. Using Minkowski inequality [9] and equalities (5), (7), we obtain:

$$\delta_n(k)=\|U_n(k(.,y);x)-k(x,y)\|_{L^2_p}=$$
$$=\frac{1}{\pi}\left\|\int_0^{2\pi}[k(x-t,y)-k(x,y)]U_n(t)dt\right\|_{L^2_p}=$$
$$=\frac{1}{\pi}\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)\left[\int_0^{2\pi}[k(x-t,y)-k(x,y)]U_n(t)dt\right]^2dx\,dy\right]^{\frac{1}{2}}\le$$
$$\le 2\,\omega_{L^2_p}(k;\frac{1}{n})\int_0^{2\pi}|U_n(t)|(n|t|+1)dt.$$

Definition 3.
We define the error of approximation of $k(x,y)$ as follows:

$$E^*_{n,\infty}(k)_{L^2_p}=\left\|k(x,y)-T^*_{n,\infty}(x,y)\right\|_{L^2_p}=$$
$$=\inf_{T_{n,\infty}(x,y)}\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[k(x,y)-T_{n,\infty}(x,y)]^2dxdy\right]^{\frac{1}{2}},$$
$$E^*_{\infty,m}(k)_{L^2_p}=\left\|k(x,y)-T^*_{\infty,m}(x,y)\right\|_{L^2_p}=$$
$$=\inf_{T_{\infty,m}(x,y)}\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[k(x,y)-T_{\infty,m}(x,y)]^2dxdy\right]^{\frac{1}{2}},$$

where $T^*_{n,\infty}(x,y)$ denotes the trigonometric polynomial in x of order n of best approximation of $k(x,y)$ in the metric L^2_p. Similarly, $T^*_{\infty,m}(x,y)$ denotes the trigonometric polynomial in y of order m of best approximation of $k(x, y)$ in the metric L^2_p. The estimates of how rapidly the quantities $E^*_{n,\infty}(k)_{L^2_p}$ and $E^*_{\infty,m}(k)_{L^2_p}$ tend to zero as $n \to \infty$; $m \to \infty$ are given in [10], where

$$E^*_{n,\infty}(k)_{L^2_p} \to 0 \qquad as\ n \to \infty. \tag{10}$$

$$E^*_{\infty,m}(k)_{L^2_p} \to 0 \qquad as\ m \to \infty \tag{11}$$

Now, we will mention bounds of the norm (8) for various linear polynomial operators U_n.

1. In the case of Jackson's method [2], [3], [9]: $U_n = J_n$

$$\delta(k;J_n) \le 12\pi\,\omega_{L^2_p}(\frac{1}{n}) \tag{12}$$

from definition (3), then:

$$E^*_{n,\infty}(k)_{L^2_p} \le \|k(x,y) - J_n[k(.,y);x]\|_{L^2_p} \le$$

$$\le 12\pi\,\omega_{L^2_p}(\frac{1}{n}) \quad \to 0 \quad as \quad n \to \infty \tag{13}$$

2. For the Vallee-Poussin's method [2], [3], [9]: $U_n = V_n$,

$$\frac{1}{\pi}\int_0^{2\pi}|V_n(t)|\,dt < \frac{1}{3} + \frac{2\sqrt{3}}{\pi} \approx 1.436$$

Considering that the method of Vallee-Poussin's V_n leaves trigonometric polynomials of order n invariant, then $V_n[T^*_{n,\infty}(.,y);x] = T^*_{n,\infty}(x,y)$ and

$$\delta(k;V_n) =$$

$$= \|(k(x,y) - T^*_{n,\infty}(x,y) - V_n[k(.,y) - T^*_{n,\infty}(.,y);x]\|_{L^2_p} \le$$

$$\le E^*_{n,\infty}(k)_{L^2_p} + \frac{1}{\pi}\int_0^{2\pi}|V_n(t)|\,.$$

$$\cdot\left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)[k(x-t,y) - T^*_{n,\infty}(x-t,y)]^2\,dydx\right]^{\frac{1}{2}} dt \le$$

$$\le\left[1 + \frac{1}{\pi}\int_0^{2\pi}|V_n(t)|\,dt\right]E^*_{n,\infty}(k)_{L^2_p} \le$$

$$\le 2.5E^*_{n,\infty}(k)_{L^2_p} \le 29.232\pi\,\omega_{L^2_p}(\frac{1}{n}) \tag{14}$$

3. In the case of Dirichlet's method [2], [3], [9] : $U_n = D_n$, by considerations similar to those used in the case of Vallee-Poussin's method and after some calculations we obtain:

$$\delta(k;D_n) \le (3+\ln n)E^*_{n,\infty}(k)_{L^2_p} \le 12\pi(3+\ln n)\,\omega_{L^2_p}(\frac{1}{n}) \tag{15}$$

4. In the case of Rogosinski's method [2], [3], [9] : $U_n = R_n$, we give two estimates. The first estimate easily follows from equation (11), we obtain :

$$\delta(k;R_n) \le (4\pi + 2\pi^2 + \frac{\pi^3}{2}\ln 2n)\,\omega_{L^2_p}(\frac{1}{n}) \tag{16}$$

In the second estimate, we let $n' = \sqrt{n}/2$, and let $a_k(y)$, $b_k(y)$ and $a^*_k(y), b^*_k(y)$ denote the corresponding coefficients of Fourier series in the variable x of the functions $k(x, y)$ and $V_{n'}[k(., y); x]$. Then, by using the Parseval's identity

$$\frac{a_0^{*2}(y)}{2} + \sum_{k=1}^{2n'}[a_k^{*2}(y) + b_k^{*2}(y)] \le \frac{1}{\pi}\int_0^{2\pi} p(x)|k(x,y)|^2 dx.$$

Therefore,

$$\left\|\frac{a_0^{*2}(y)}{2} + \sum_{k=1}^{2n'}[a_k^{*2}(y) + b_k^{*2}(y)]^{\frac{1}{2}}\right\|_{L^2_p} \le \frac{1}{\sqrt{\pi}}\|k(x,y)\|_{L^2_p}.$$

On the other hand, from (14) we have:

$$\delta(k;R_n) = \|k(x,y) - V_{n'}(k(.,y);x) + V_{n'}(k(.,y);x) -$$

$$- R_n(V_{n'}(k(.,y);x)) + R_n(V_{n'} - k;x)\|_{L^2_p} \le$$

$$\le 7.5E^*_{n',\infty}(k)_{L^2_p} + \frac{\pi^{3/2}}{8}n^{-3/4}\|k(x,y)\|_{L^2_p} \tag{17}$$

5. In the case of Féjer's method [2], [3], [9] : $U_n = F_n$, by consideration similar to those used in the case of Rogozinski's method we fined :

$$\delta(k;F_n) \le 5E^*_{n',\infty}(k)_{L^2_p} + \frac{1}{\sqrt{\pi}}n^{-1/4}\|k(x,y)\|_{L^2_p} \le$$

$$\le 12(1+\ln n)\,\omega_{L^2_p}(\frac{1}{n}) \tag{18}$$

Now, it is clear that $\delta(k,U_n) \to 0$ as $n \to \infty$ for Jackson's, Vallee-Poussin's, Féjer's and Rogozinski's methods for every square integrable periodic function $k(x, y) \in \boldsymbol{L^2_p}$. In the case of Dirichlet's method $\delta(k,D_n) \to 0$ as $n \to \infty$ if $\omega_{L^2_p}(\frac{1}{n}) = o(1/\ln n)$ or $E^*_{n,\infty}(k)_{L^2_p} = o(1/\ln n)$.

Definition 4.

The following quantities will play an important role in estimating the error of our approximation:

$$\zeta_n = \zeta(k;U_n;\varphi) = \left\|\int_0^{2\pi} k(x,y)[U_n(\varphi;y) - \varphi(y)]dy\right\|_{L^2_p}; \tag{19}$$

and
$$\gamma_m = \gamma_m(U_n;\varphi) = \sum_{i=1}^{m}\left|1-\lambda_i^{(n)}\right| E_{i-1}(\varphi)_{L^2_p}. \tag{20}$$

where

$$E_n(\varphi)_{L^2_p} = \inf_{T_n}\|\varphi(x) - T_n(x)\|_{L^2_p},$$

$T_n(x)$ is a trigonometric polynomial of order n in x, $m \le n$.

Let us denote by $W^rH^\beta(L^2_p)$ (r-non negative integer, $0 < \beta \le 1$) the class of periodic functions of period 2π possessing the derivatives of order r, which satisfies Lipschitz condition of order β [3], [10] and denote to:

$$\gamma_m(U_n;W^rH^\beta(L^2_p)) = \sum_{i=1}^{m}\left|1-\lambda_i^{(n)}\right| \sup_{f\in W^rH^\beta(L^2_p)} E_{i-1}(f)_{L^2_p} \le$$

$$\le \frac{1}{4}\pi^{1+\beta}\sum_{i=1}^{m}\left(\left|1-\lambda_i^{(n)}\right|/i^{r+\beta}\right). \tag{21}$$

We shall derive from this inequality the following condition

$$\gamma_m(U_n;W^rH^\beta(L_p^2))=o\left\{\sup_{\varphi\in W^rH^\beta(L_p^2)}\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2}\right\} \qquad (22)$$

It is clear that

$$\gamma_m(D_n;W^rH^\beta(L_p^2))=\gamma_m(V_n;W^rH^\beta(L_p^2))=0$$

for all $m\le n$.

Then condition (22) will be satisfied for all r = 0, 1, 2,…, $0<\beta\le 1$ in Dirichlet's and Valle-Poussin's methods. In the case of Féjer's method, condition (22) is valid only for $r=0$, $0<\beta\le 1$. But in the case of Jackson's and Rogozinski's method, condition (22) shall be valid if $r+\beta<2$.

Theorem 5.

For any square-summable kernel $k(x, y)\in L^2_p$ and for linear polynomial operator $U_n(g; x)$, the following inequality holds:

$$\zeta_n=\zeta(k;U_n;\varphi)=\left\|\int_0^{2\pi}k(x,y)[U_n(\varphi;y)-\varphi(y)]dy\right\|_{L_p^2}$$

$$\le\left[E^*_{\infty,m}(k)_{L_p^2}\|\varphi(y)-U_n(\varphi;y)\|_{L_p^2}+\right.$$

$$\left.+\sqrt{\frac{2}{\pi}}\gamma_m(U_n;\varphi)\left[\int_0^{2\pi}p(x)dx\right]^{\frac{1}{2}}\left[\|k(x,y)\|_{L_p^2}+E^*_{\infty,m}(k)_{L_p^2}\right]\right] \qquad (23)$$

for any positive integer $m\le n$.

Proof. For any function $\varphi(x)\in L^2_p$ with Fourier coefficients c_i and d_i, and because of $p(x)\ge 1$, the following inequality holds:

$$|c_i\cos(ix)+d_i\sin(ix)|=$$

$$=\inf_{T_{i-1}(t)}\frac{1}{\pi}\left|\int_0^{2\pi}[\varphi(t)-T_{i-1}(t)]\cos(i(x-t))dt\right|\le$$

$$\le\frac{1}{\pi}\inf_{T_{i-1}(t)}\left[\int_0^{2\pi}p(t)[\varphi(t)-T_{i-1}(t)]^2dt\right]^{\frac{1}{2}}\cdot$$

$$\cdot\left[\int_0^{2\pi}\frac{[\cos(i(x-t))]^2}{p(t)}dt\right]^{\frac{1}{2}}\le\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2}$$

therefore

$$\|c_i\cos(ix)+d_i\sin(ix)\|_{L_p^2}\le\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2}\left(\int_0^{2\pi}p(x)dx\right)^{\frac{1}{2}}$$

Letting

$$T_{\infty,m}(x,y)=\sum_{i=0}^m a_i(x)\cos(iy)+b_i(x)\sin(iy),$$

$$E^*_{\infty,m}(k)_{L_p^2}=\inf_{a_i,b_i}\left\|k(x,y)-\sum_{i=0}^m a_i(x)\cos(iy)+b_i(x)\sin(iy)\right\|_{L_p^2},$$

and taking into consideration (20) and using Bunyakovskii inequality, we obtain :

$$\zeta_n=\zeta(k;U_n;\varphi)=\left\|\int_0^{2\pi}k(x,y)[U_n(\varphi;y)-\varphi(y)]dy\right\|_{L_p^2}=$$

$$=\left[\int_0^{2\pi}p(x)\left[\int_0^{2\pi}k(x,y)[U_n(\varphi;y)-\varphi(y)]dy\right]^2dx\right]^{\frac{1}{2}}\le$$

$$\le\left[\int_0^{2\pi}p(x)\inf_{T_{\infty,m}(x,y)}\left[\int_0^{2\pi}|k(x,y)-T_{\infty,m}(x,y)||\varphi(y)-U_n(\varphi,y)|dy+\right.\right.$$

$$\left.\left.+\left|\int_0^{2\pi}(k(x,y)+T_{\infty,m}(x,y)-k(x,y))(\varphi(y)-U_n(\varphi;y))dy\right|\right]^2dx\right]^{\frac{1}{2}}\le$$

i.e. $\zeta_n\le$

$$\le\left[\int_0^{2\pi}p(x)\inf_{T_{\infty,m}(x,y)}\left[\int_0^{2\pi}|k(x,y)-T_{\infty,m}(x,y)||\varphi(y)-U_n(\varphi;y)|dy\right]^2dx\right]^{\frac{1}{2}}+$$

$$+\left[\int_0^{2\pi}p(x)\inf_{T_{\infty,m}(x,y)}\left[\int_0^{2\pi}(k(x,y)+T_{\infty,m}(x,y)-k(x,y))\cdot\right.\right.$$

$$\left.\left.\cdot\left(\sum_{i=0}^m(1-\lambda_i^{(n)})(c_i\cos(iy)+d_i\sin(iy))\right)dy\right]^2dx\right]^{\frac{1}{2}}\le$$

$$\le E^*_{\infty,m}(k)_{L_p^2}\|\varphi(y)-U_n(\varphi;y)\|_{L_p^2}+$$

$$+\sqrt{\frac{2}{\pi}}\gamma_m(U_n;\varphi)\left(\int_0^{2\pi}p(x)dx\right)^{\frac{1}{2}}\left[\|k(x,y)\|_{L_p^2}+E^*_{\infty,m}(k)_{L_p^2}\right]$$

The estimates of how rapidly the quantities $E^*_{\infty,m}(k)_{L_p^2}$ and $\gamma_m(U_n;\varphi)$, tends to zero as $m\to\infty$, are given in [3], [10].

IV. THE APPROXIMATE SOLUTION

The following theorem shows that for sufficiently good linear methods $U_n(g; x)$, the difference between the polynomials $\varphi_n(x)$ and the original solution $\varphi(x)$ is sufficiently small.

Theorem 6.

If the kernel $k(x, y)$ of (1) satisfies the assumptions (A^*), all functions appearing in (1) are 2π – periodic in x and y, then for any linear polynomial operator $U_n(g; x)$, if $|\lambda|R\delta(k; U_n)<1$ and if equation (1) is replaced by equation (4), the following inequality holds:

$$\|\varphi(x)-\varphi_n(x)\|_{L_p^2}\le(1+\alpha_n(k))\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2} \qquad (24)$$

in which

$$\alpha_n(k)=|\lambda|R\left[\delta(k;U_n)+\frac{\zeta(k;U_n;\varphi)}{\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2}}\right]/\left[1-|\lambda|R\delta(k;U_n)\right]. \qquad (25)$$

where $\delta(k; U_n)$ and $\zeta(k; U_n; \varphi)$ are defined from (8)and (19) respectively, and $R=(1+|\lambda|\|R(x,y)\|_{L_p^2})$, where $R(x, y)$ denotes the resolvent of the kernel $k(x, y)$.

Proof. By using equation (1), we represent the solution $\varphi_n(x)$ of (4) in the form :

$$\varphi_n(x)=U_n(f;x)+\lambda U_n\left[\int_0^{2\pi}k(.,y)[\varphi_n(y)-\varphi(y)]dy+\right.$$

$$\left.+\int_0^{2\pi}k(.,y)\varphi(y)dy;x\right]=\lambda\int_0^{2\pi}U_n[k(.,y);x][\varphi_n(y)-\varphi(y)]dy+$$

$$+U_n\left[f(.)+\lambda\int_0^{2\pi}k(.,y)\varphi(y)dy;x\right]=$$

$$=\lambda\int_0^{2\pi}U_n[k(.,y);x][\varphi_n(y)-\varphi(y)]dy+U_n(\varphi;x), \qquad (26)$$

whence, it follows that:

$$\varphi_n(x)-U_n(\varphi;x)=\lambda\int_0^{2\pi}k(x,y)[\varphi_n(y)-U_n(\varphi;y)]dy+g_n(x), \qquad (27)$$

where

$$g_n(x)=\lambda\int_0^{2\pi}[U_n(k(.,y);x)-k(x,y)][\varphi_n(y)-\varphi(y)]dy+$$
$$+\lambda\int_0^{2\pi}k(x,y)[U_n(\varphi;y)-\varphi(y)]dy.$$

Thus, by (8), (9) and (19), we have the estimate

$$\|g_n(x)\|_{L_p^2}\le|\lambda|\delta(k;U_n)\Big[\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}+$$
$$+\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}\Big]+|\lambda|\zeta(k;U_n;\varphi). \qquad (28)$$

In view of theorem 2. Equation (27) has a unique solution given by:

$$\varphi_n(x)-U_n(\varphi;x)=g_n(x)+\lambda\int_0^{2\pi}R(x,y)g_n(y)dy.$$

Therefore;

$$\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}\le\|g_n(x)\|_{L_p^2}\Big[1+|\lambda|\|R(x,y)\|_{L_p^2}\Big]=$$
$$=R\|g_n(x)\|_{L_p^2}\le R|\lambda|\Big[\delta(k;U_n)\Big[\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}+$$
$$+\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}\Big]+\zeta(k;U_n;\varphi)\Big].$$

Taking into consideration $|\lambda| R\ \delta(k;\ U_n) < 1$ and after some calculations, we obtain:

$$\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}\le$$
$$\le\frac{|\lambda|R\Big[\delta(k;U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}+\zeta(k;U_n;\varphi)\Big]}{1-|\lambda|R\,\delta(k;U_n)}$$

Therefore;

$$\|\varphi(x)-\varphi_n(x)\|_{L_p^2}\le\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2}+\|\varphi_n(x)-U_n(\varphi;x)\|_{L_p^2}\le$$
$$\le\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2}+$$
$$+\frac{|\lambda|R\Big[\delta(k;U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}+\zeta(k;U_n;\varphi)\Big]}{1-|\lambda|R\,\delta(k;U_n)}\le$$
$$\le(1+\alpha_n(k))\|\varphi(x)-U_n(\varphi;x)\|_{L_p^2},$$

where $\alpha_n(k)$ is given by (25). Thus, inequality (24) is proved.

V. CONCLUSIONS

In conclusion we note the following statements :

1) It is well-known [5] that one cannot achieve an error less than that corresponding to the best approximation . The error estimate in (24) with rate of convergence $\alpha_n(k)$, means that: the rate of convergence of $\varphi_n(x)$ to $\varphi(x)$ is comparable with the rate of convergence of the best approximation , which means that the error estimate in (24) is optimal .

2) Applying theorem 6, and also the corresponding results from section 3, we obtain the following results:

I) In the case of the application of Dirichlet's method, from (15) and (24), we have:

$$\|\varphi(x)-\varphi_n(x)\|_{L_p^2}\le(1+\alpha_n(k))(3+\ln n)E_n^*(\varphi)_{L^2_{p(x)}}$$

where $E_n^*(\varphi)_{L^2_{p(x)}}$ is the best approximation to $\varphi(x)$ by trigonometric polynomial and depends on the smoothness of $\varphi(x)$ which in turn depends on the smoothness of $f(x)$ and $k(x, y)$. As a consequence of (15), (21), (22), (23) and (25), we obtain the following estimate for $\alpha_n(k)$:

$$\alpha_n(k)\le|\lambda|R\frac{(3+\ln n)E^*_{n,\infty}(k)_{L^2_{p(x)}}+E^*_{\infty,n}(k)_{L^2_{p(x)}}}{1-|\lambda|R(3+\ln n)E^*_{n,\infty}(k)_{L^2_{p(x)}}}\le$$
$$\le|\lambda|R\frac{12\pi(3+\ln n)\,\omega_{L_p^2}(1/n)+E^*_{\infty,n}(k)_{L^2_{p(x)}}}{1-12\pi|\lambda|R(3+\ln n)\,\omega_{L_p^2}(1/n)}$$

$\alpha_n(k)\to 0$ as $n\to\infty$ for $\varphi(x)\in L^2_{p(x)}$, $k(x, y)\in L^2_p[Q]$ whenever $\omega_{L_p^2}(1/n)=o(1/\ln n)$ or $E^*_{n,\infty}(k)_{L_p^2}=o(1/\ln n)$.

II) In the case of Vallee-Poisson's method, considering (21) , (22) and (24), we obtain:

$$\|\varphi(x)-\varphi_n(x)\|_{L^2_{p(x)}}\le(1+\alpha_n(k))\Big(\frac{4}{3}+\frac{2\sqrt{3}}{\pi}\Big)E_n^*(\varphi)_{L^2_{p(x)}}\le$$
$$\le(1+\alpha_n(k))(2.5)E_n^*(\varphi)_{L^2_{p(x)}},$$

where by (14)

$$\alpha_n(k)\le|\lambda|R\frac{2.5E^*_{n,\infty}(k)_{L^2_{p(x)}}+E^*_{\infty,n}(k)_{L^2_{p(x)}}}{1-|\lambda|R(2.5)E^*_{n,\infty}(k)_{L^2_{p(x)}}}$$

then, $\alpha_n(k)\to 0$ as $n\to\infty$ for $\varphi(x)\in L^2_{p(x)}$, $k(x, y)\in L^2_p[Q]$

III) For the methods of Féjer, Jackson and Rogosinski, the quantity $\alpha_n(k)$ in the relation (24) will not tend to zero for any solution $\varphi(x)$, but will tend to zero only under the condition that "the solution $\varphi(x)$, belongs to some subclasses of integrable functions". Restricting ourselves to the holder classes $W^rH^\beta(L_p^2)$ where r is a non-negative integer and $0 < \beta \le 1$, we obtain the following two cases :

a) In the case of Féjer's method; in order to $\alpha_n(k)\to 0$ as $n\to\infty$ considering (18), (21) and (22), it is sufficient that the following conditions be satisfied :

$$\varphi(x)\in W^0H^\beta(L^2_{p(x)})\text{ i.e. } r=0,\ 0<\beta\le 1,\quad \omega_{L_p^2}(1/n)=o(1/\ln n)$$

b) For Rogosinski's and Jackson's methods; in order that $\alpha_n(k)\to 0$ as $n\to\infty$, using (12) , (17), (21) and (22), it is then sufficient that the following conditions be satisfied :

$$\varphi(x)\in W^rH^\beta(L^2_{p(x)}),\qquad r+\beta<2,\qquad 0<\beta\le 1.$$

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